Multi-graviton theory, a latticized dimension, and the cosmological constant

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Abstract

Beginning with the Pauli-Fierz theory, we construct a model for multi-graviton theory. Couplings between gravitons belonging to nearest-neighbor "theory spaces" lead to a discrete mass spectrum. Our model coincides with the Kaluza-Klein theory whose fifth dimension is latticized.

We evaluate one-loop vacuum energy in models with a circular latticized extra dimension as well as with compact continuous dimensions. We find that the vacuum energy can take a positive value, if the dimension of the continuous space time is $6, 10, \ldots$. Moreover, since the amount of the vacuum energy can be an arbitrary small value according to the choice of parameters in the model, our models is useful to explain the small positive dark energy in the present universe.

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I. INTRODUCTION AND SUMMARY

It is well known that the weak-field limit of the Einstein gravity reduces to the theory of a single massless spin-two field [1]. We call the second-order symmetric tensor field $h_{\mu\nu}$, which represents deviation from a flat space ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), as a "graviton". Multi-graviton theories have been studied by Boulanger *et al.* [2, 3]. They have shown that there is no consistent interaction of gravitons and the possible action is a sum of replicated Pauli-Fierz actions [4].

In this paper, we consider a model for a multi-graviton theory with nearest-neighbor couplings in the theory space. To be precise, the gravitons are not interacting each other in our model, but they have a discrete mass spectrum. The simplest model is explained in Sec. II. In the limit of a large number of gravitons, the mass spectrum of gravitons resembles that of the Kaluza-Klein (KK) theory. Therefore our model is equivalent to the KK theory with a discretized dimension.

The emergence mechanisms of a space dimension have been suggested in refs. [5, 6, 7], which is in the stream of dimensional deconstruction [8, 9]. Recently the authors have been informed of ref. [10], in which the gravity with the theory space is also argued. Although our model for the gravitation theory with a discretized dimension is very simple, it may be a toy model adequate for studying qualitative features of more complicated theory and mechanisms included in it.

For example, we evaluate the vacuum energy in our model in Sec. III. We also consider a model in the continuous $(4 + \delta)$ dimensional space-time with a discretized dimension. We show that an arbitrary small amount of positive vacuum energy can be obtained if $(4 + \delta) = 6, 10, \ldots$.

II. MULTI-GRAVITON THEORY AND A LATTICIZED DIMENSION

The lagrangian for a massless spin-two field can be written as

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\nu\mu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h \,, \tag{1}$$

where $h \equiv h^\mu_\mu$. This corresponds with the weak-field limit of the Einstein gravity.

The lagrangian for a massive graviton with the Stückerberg fields [11] is known as [12]

$$\mathcal{L}_m = \mathcal{L}_0 - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2) - 2(mA^\mu + \partial^\mu\varphi)(\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2, \quad (2)$$

where a constant m corresponds to the mass of graviton. We have omitted here the gauge fixing term as well as the ghost terms. The lagrangian (2) is invariant under the following transformations:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad (3)$$

$$A_\mu \rightarrow A_\mu + m\xi_\mu - \partial_\mu\zeta, \quad (4)$$

$$\varphi \rightarrow \varphi + m\zeta, \quad (5)$$

where ξ and ζ are local parameters.

Next we consider a multi-graviton (N-graviton) theory [2, 3] given by the lagrangian

$$\bar{\mathcal{L}}_0 = \sum_{k=1}^{N} \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}^k \partial^\lambda h^{k\,\mu\nu} + \partial_\lambda h^{k\,\lambda}{}_\mu \partial_\nu h^{k\,\nu\mu} - \partial_\mu h^{k\,\mu\nu}\partial_\nu h^k + \frac{1}{2}\partial_\lambda h^k \partial^\lambda h^k \right]. \quad (6)$$

It can be said that a massless graviton lives in each "theory space", which is labeled by k.

Now, instead of adding the mere mass term to the lagrangian (6), we put the term including couplings of other gravitons with the nearest-neighbor suffix:

$$\bar{\mathcal{L}}_m = \bar{\mathcal{L}}_0 - \frac{m^2}{2}\sum_{k=1}^{N}\left[(h_{\mu\nu}^k - h_{\mu\nu}^{k+1})^2 - (h^k - h^{k+1})^2\right]$$

$$-2\sum_{k=1}^{N}\left[m(A^{k\,\mu} - A^{k-1\,\mu}) + \partial^\mu\varphi^k\right](\partial^\nu h_{\mu\nu}^k - \partial_\mu h^k) - \frac{1}{2}\sum_{k=1}^{N}(\partial_\mu A_\nu^k - \partial_\nu A_\mu^k)^2. \quad (7)$$

In this notation, one should read as $h^{k+N} = h^k$, $h^0 = h^N$, $h^{N+1} = h^1$, and so on.

It is interesting to see that the multi-graviton lagrangian is invariant under the transformations:

$$h_{\mu\nu}^k \rightarrow h_{\mu\nu}^k + \partial_\mu\xi_\nu^k + \partial_\nu\xi_\mu^k,$$

$$A_\mu^k \rightarrow A_\mu^k + m\xi_\mu^k - m\xi_\mu^{k+1} - \partial_\mu\zeta^k,$$

$$\varphi^k \rightarrow \varphi^k + m\zeta^k - m\zeta^{k-1}. \quad (8)$$

As in the case with 5D QED studied by Hill and Leibovich [13], $N-1$ massive gravitons and one massless graviton are described by the lagrangian. The eigenvalues for the mass matrix are

$$M_p^2 = 4m^2\sin^2\left(\frac{\pi p}{N}\right) \qquad (p = 1, \ldots, N). \quad (9)$$

Massive vector and scalar fields are absorbed by the massive graviton fields by means of the transformations (8) and then they have correct spin-degree of freedom, five.

On the other hand, massless vector and scalar fields are survived. The massless degree of freedom of the spin-two, one, zero fields are described as

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} h_{\mu\nu}^{k} , \qquad \bar{A}_{\mu} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} A_{\mu}^{k} , \qquad \bar{\varphi} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varphi^{k} . \tag{10}$$

When we make the following combination

$$H_{\mu\nu} = \bar{h}_{\mu\nu} + \bar{\varphi}\eta_{\mu\nu} , \tag{11}$$

it becomes a massless graviton and then the kinetic term of $\bar{\varphi}$ takes a canonical form [12].

We find that the field content of our model is very akin to the one of the KK theory [14]. Suppose that we considered the five-dimensional metric as

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu} - 2A_{\mu}dx^{\mu}dy + (1 + 2\varphi)dy^2 , \tag{12}$$

or

$$A_{\mu} = h_{\mu y} , \qquad \varphi = h_{yy} . \tag{13}$$

When we take the weak-field limit of the five-dimensional Einstein-Hilbert action, the lagrangian density appears to be of the form (7), provided that the derivative with respect to the extra dimensional coordinate y is replaced by the difference operation, namely

$$\frac{\partial h_{\mu\nu}}{\partial y} \longrightarrow \Delta_y h_{\mu\nu} \equiv \frac{h_{\mu\nu}(x, y+\epsilon) - h_{\mu\nu}(x, y)}{\epsilon} , \tag{14}$$

and we define $h_{\mu\nu}^{k}(x) \equiv h_{\mu\nu}(x, k\epsilon)$ with $\epsilon = m^{-1}$.

The mass spectrum and spin degrees of freedom in the KK theory have been explained by ref. [15]. Our multi-graviton model can be regarded as a discrete version of the KK theory. Thus the transformation (8) can also be derived in the similar manner to ref. [15].

The simplest way to get actions for multi-graviton system is the discretization of extra dimensions in generalized KK theories. The counting of the four-dimensional fields can be done by the completely parallel way to the analysis in ref. [16, 17]. If we consider the discretized version of the KK theory in $M_4 \otimes T^d$ (where M_4 denotes the four-dimensional Minkowsky spacetime and T^d is a d dimensional torus), we obtain a tower of massive spin-two states, $(d-1)$ tower of massive spin-one states, and $d(d-1)/2$ tower of massive spinless

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states. As well, we obtain the following massless states: one spin-two, d spin-one, and $d(d+1)/2$ spinless states.

It is an important problem whether all of the field contents are necessary or not in the view of the four-dimensional gravity. In other words, we are interested in the possibility of the multi-graviton theory whose continuum limit does not coincide with a higher-dimensional theory. Particularly, the presence of the massless vector and scalar fields are unwanted when we wish to consider a phenomenological model. Attempts to solve this problem by imposing discrete symmetries in the theory space and discretization of non-abelian KK theories will be considered elsewhere.

III. VACUUM ENERGY

In this section, we evaluate the one-loop vacuum energy density in the model of the previous section. We will also offer an extension of the model, which includes both latticized and continuous extra dimensions, and the vacuum energy is estimated in the model.

Recently, the scenarios explaining the small positive cosmological constant in the present universe ($10^{-47}\mathrm{GeV}^4$, according to [18]) are discussed through the study of the Casimir-like energy in the higher-dimensional theories [19, 20]. Because the latticized model would have different UV behavior from that of continuum model, we expect a novel expression of the cosmological constant.

First, we consider one scalar degree of freedom which has the mass spectrum discussed in the previous section. Then the vacuum energy density in the one-loop calculation is written by the zeta-function technique [21] as

$$\frac{1}{2} \ln \det[-\nabla^2 + M_p^2] = -\frac{1}{2} \left. \frac{d\zeta(s)}{ds} \right|_{s=0}, \tag{15}$$

where

$$\zeta(s) = \frac{\mu^{2s}}{16\pi^2 \Gamma(s)} \sum_p \int_0^\infty dt \, t^{s-3} \exp(-M_p^2 t), \tag{16}$$

with

$$M_p^2 = 4m^2 \sin^2\left(\frac{\pi p}{N}\right). \tag{17}$$

Here the constant μ has the dimension of mass.

The calculation of the zeta function can be done by the similar method used in refs. [22, 23]. Using the formula

$$\exp\left[-4m^2\sin^2(\theta/2)t\right] = e^{-2m^2t}\sum_{\ell=-\infty}^{\infty}\cos\ell\theta \; I_\ell(2m^2t),\tag{18}$$

where $I_\nu(x)$ is the modified Bessel function, we can write the zeta function as

$$\zeta(s) = \frac{\mu^{2s}}{(4\pi)^2\Gamma(s)}\sum_p\sum_{\ell=-\infty}^{\infty}\cos\frac{2\pi p\ell}{N}\int_0^\infty dt\; t^{s-3}e^{-2m^2t}I_\ell(2m^2t).\tag{19}$$

For simplicity, we assume $N \geq 3$. Carrying out the summation over p first, we find that only the terms of $\ell = qN$ ($q: integer$) survive. Then we find

$$\zeta(s) = \frac{\mu^{2s}N}{(4\pi)^2\Gamma(s)}\left[\int_0^\infty dt\; t^{s-3}e^{-2m^2t}I_0(2m^2t) + 2\sum_{q=1}^{\infty}\int_0^\infty dt\; t^{s-3}e^{-2m^2t}I_{qN}(2m^2t)\right].\tag{20}$$

According to ref. [24], the integration can be carried out and is found to be

$$\int_0^\infty dt\; t^{s-3}e^{-2m^2t}I_\ell(2m^2t) = (4m^2)^{2-s}\frac{\Gamma(\frac{5}{2}-s)\Gamma(\ell-2+s)}{\sqrt{\pi}\Gamma(\ell+3-s)}.\tag{21}$$

Therefore the zeta function is written as

$$\zeta(s) = \frac{N\,m^4}{\pi^2}\left(\frac{\mu^2}{4m^2}\right)^s\frac{\Gamma(\frac{5}{2}-s)}{\sqrt{\pi}}\left[\frac{1}{(s-1)(s-2)\Gamma(3-s)} + \frac{2}{\Gamma(s)}\sum_{q=1}^{\infty}\frac{\Gamma(qN-2+s)}{\Gamma(qN+3-s)}\right].\tag{22}$$

Now we obtain

$$-\left.\frac{d\zeta(s)}{ds}\right|_{s=0} = -\frac{3N\,m^4}{16\pi^2}\ln\left(\frac{\bar{\mu}^2}{4m^2}\right) - \frac{3m^4}{2\pi^2}\sum_{q=1}^{\infty}\frac{1}{q(q^2N^2-1)(q^2N^2-4)},\tag{23}$$

where

$$\ln\left(\frac{\bar{\mu}^2}{4m^2}\right) = \ln\left(\frac{\mu^2}{4m^2}\right) + \psi(3) - \psi(5/2) + \frac{3}{2}.\tag{24}$$

Because there are five states in each mass level, the vacuum energy in the multi-graviton model in the previous section is

$$V = V_R - \frac{15N\,m^4}{32\pi^2}\ln\left(\frac{\bar{\mu}^2}{4m^2}\right) - \frac{15m^4}{4\pi^2}\sum_{q=1}^{\infty}\frac{1}{q(q^2N^2-1)(q^2N^2-4)},\tag{25}$$

where $V_R(\bar{\mu})$ is the renormalized vacuum energy.

According to ref. [25], we require the invariance of V with respect to the scale $\bar{\mu}$ and we choose $V_R(\bar{\mu}_R) = 0$ for a normalization point $\bar{\mu}_R$. Then we find

$$V = -\frac{15N\,m^4}{32\pi^2}\ln\left(\frac{\bar{\mu}_R^2}{4m^2}\right) - \frac{15m^4}{4\pi^2}\sum_{q=1}^{\infty}\frac{1}{q(q^2N^2-1)(q^2N^2-4)}.\tag{26}$$

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Unfortunately, this vacuum energy density is negative, for $\bar{\mu}_R$ is considered as a cutoff due to a scale of a new physics, $\bar{\mu}_R > m$. The boundary condition $V_R(\bar{\mu}_R) = 0$ can be interpreted as the expectation that the new physics forces the vacuum energy or cosmological constant to vanish at such a high-energy scale.

Let us consider the extension of our model to the one with $4 + \delta$ dimensional continuous spacetime. Here the extra δ dimensional space is taken as $(S^1)^\delta$ with a common radius b. One can regard the new model as the theory with a latticized circle and δ continuous circles. In this model the mass spectrum of the gravitons is expressed as

$$M_{p,\mathbf{n}}^2 = 4m^2 \sin^2\left(\frac{\pi p}{N}\right) + \frac{\mathbf{n}^2}{b^2}, \tag{27}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_\delta)$ and all the components take integer values.

Using the Poisson resummation formula

$$\sum_{\mathbf{n}} \exp\left(-\frac{\mathbf{n}^2}{b^2}t\right) = \left(\frac{\pi b^2}{t}\right)^{\delta/2} \sum_{\mathbf{n}} \exp\left(-\frac{\pi^2 b^2 \mathbf{n}^2}{t}\right), \tag{28}$$

we can express the zeta function, in which in the present time the summation over $\mathbf{n}$ is involved, as

$$\begin{aligned}
\zeta(s) = & \frac{\mu^{2s} N (2\pi b)^\delta}{(4\pi)^{2+\delta/2} \Gamma(s)} \sum_{\mathbf{n}} \left[\int_0^\infty dt\, t^{s-\delta/2-3} \exp\left(-2m^2 t - \pi^2 b^2 \mathbf{n}^2/t\right) I_0(2m^2 t) \right. \\
& \left. + 2 \sum_{q=1}^\infty \int_0^\infty dt\, t^{s-\delta/2-3} \exp\left(-2m^2 t - \pi^2 b^2 \mathbf{n}^2/t\right) I_{qN}(2m^2 t) \right].
\end{aligned} \tag{29}$$

If δ is odd, all the terms in the parentheses are finite and lead to no logarithm term in the vacuum energy. The vacuum energy can be found to be finite and negative in this case, as in the case of the KK theory with torus compactification. Thus we assume that δ is even.

For even δ, in the limit $s \to 0$, only the first term with $\mathbf{n} = \mathbf{0}$ is divergent in the parentheses on the right hand side of (29) if N is larger than $2 + \delta/2$. Thus, after taking account of the degrees of freedom, the vacuum energy density in our extended model takes the form

$$V = V_R - (-1)^{\delta/2} \frac{(5+\delta)(2+\delta)N\, m^4}{4\pi^2} (4\pi m^2 b^2)^{\delta/2} \frac{\Gamma(\frac{5+\delta}{2})}{\sqrt{\pi}\Gamma(3+\frac{\delta}{2})^2} \ln\left(\frac{\bar{\mu}^2}{4m^2}\right) - \cdots, \tag{30}$$

where

$$\ln\left(\frac{\bar{\mu}^2}{4m^2}\right) = \ln\left(\frac{\mu^2}{4m^2}\right) + 2\psi(3+\delta/2) - \psi(5/2+\delta/2) + \gamma, \tag{31}$$

(γ is the Euler's constant) and ... means the positive finite terms.

As seen from the sign of logarithmic term, we can obtain the positive vacuum energy after taking account of the renormalization discussed above; in other words, we have the positive cosmological constant when $\delta = 4k + 2$ ($k = 0, 1, 2, \ldots$), such as $\delta = 2$ ($D = 4 + \delta = 6$), $\delta = 6$ ($D = 10$), and so on.

We will further investigate the possibility of the arbitrarily small cosmological constant. Since the contribution from the finite q terms in the zeta function can be estimated as at most $O(N^{-4})$, it can be neglected if a large N is assumed. Further if $mb \ll 1$, we find

$$
\begin{aligned}
V \approx & \frac{(5 + \delta)(2 + \delta)N\,m^4}{4\pi^2}(4\pi m^2 b^2)^{\delta/2}\frac{\Gamma(\frac{5+\delta}{2})}{\sqrt{\pi}\Gamma(3 + \frac{\delta}{2})^2}\ln\left(\frac{\bar{\mu}_R^2}{4m^2}\right) \\
& - \frac{(5 + \delta)(2 + \delta)N}{4\pi^{(4+\delta)/2}(2\pi b)^4}\Gamma\left(\frac{4 + \delta}{2}\right)\sum_{\mathbf{n}\neq\mathbf{0}}\frac{1}{(\mathbf{n}^2)^{(4+\delta)/2}} \qquad (mb \ll 1).
\end{aligned}
\tag{32}
$$

Therefore when $(2\pi mb)^{4+\delta}\ln(\bar{\mu}_R/m) \approx O(1)$, an arbitrary small vacuum energy can be obtained. Note that the condition does not include N in this approximation.

The difficulty of the present model in this section is the problem of tuning the scale of the extra dimensions. The model does not provide a mechanism to determine the value of b, because there is no minimum value of V as a function of b. We should consider either the value of b is due to the other mechanism or b takes an appropriate value only in the present universe. For the latter case, the motion of b may play a role of quintessence or dark energy, as similar to the suggestion in ref. [26]. This possibility will be considered elsewhere.

In this paper, we only considered the vacuum energy at the one-loop level. Even at the same level, the correction to the Newton constant may also be induced in general. This would be related with the restriction or determination of the scale of the extra dimensions as well as the scale of m.

We should investigate the nature of coupling to the matter fields and the back reaction problem in the case with a small cosmological constant. Through such studies, the case with strong gravity will also be revealed.

Finally, we remember that we have not treated the hierarchy problem in this paper. It will be clarified by the further study whether incorporating supersymmetry (and its break-down) into the model is necessary or not.

Acknowledgments

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